
Definitions:

Let S be a nonempty subset of $\mathbb{R}$, i.e. $\phi \neq S \subseteq \mathbb{R}$

(1) If $x_0 \in S$ and $x \leq x_0$ for all $x \in S$,
then x_0 is called the maximum of S. ($x_0 = \max S$.)

(2) If $x_0 \in S$ and $x_0 \leq x$ for all $x \in S$,
then x_0 is called the minimum of S. ($x_0 = \min S$.)

(3) If $\exists M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$,
then M is called an **upper bound** of S and the set S is **bounded above**.

(4) If $\exists m \in \mathbb{R}$ such that $m \leq x$ for all $x \in S$,
then m is called a **lower bound** of S and the set S is **bounded below**.

(5) If $\exists m, M \in \mathbb{R}$ such that $m \leq x \leq M$ $\forall x \in S$, then S is **bounded**.

(6) If S is bounded above and S has a least upper bound M_0, then M_0 is called the supremum of S and denoted by $\sup S$.

(7) If S is bounded below and S has a greatest lower bound m_0, then m_0 is called the infimum of S and denoted by $\inf S$.

The Completeness Axiom

A fundamental property of the set $\mathbb{R}$ of real numbers :

Completeness Axiom : $\mathbb{R}$ has "no gaps".

$\forall S \subseteq \mathbb{R}$ and $S \neq \emptyset$,

If S is bounded above, then $\sup S$ exists and $\sup S \in \mathbb{R}$.

(that is, the set S has a least upper bound which is a real number).

Note : "The Completeness Axiom" distinguishes the set of real numbers $\mathbb{R}$ from other sets such as the set $\mathbb{Q}$ of rational numbers.

Example: Let $A := \{r \in \mathbb{Q} | 0 \leq r \leq \sqrt{2}\} \subseteq \mathbb{Q}$.

(1) Is the set A bounded above?

(2) Does it has a least upper bound in A?

Examples: Find the inf and sup of the following sets, if possible. State whether or not these numbers are in S.

1. $S = \{x \,|\, 0 < x \leq 3\}$

2. $S = \{x \,|\, x^2 - 2x - 3 < 0\}$

3. $S = \{x \,|\, 0 < x < 5, \cos(x) = 0\}$

4. $S = \{x \,|\, x = \frac{1}{n}, n \in \mathbb{N}\}$

Some properties of sup and inf Theorem. If x_1 and x_2 are least upper bounds for the set A, then $x_1 = x_2$.

Theorem. If the sets A and B are bounded above and $A \subseteq B$, then $\sup(A) \leq \sup(B)$.